4/28/2015 - Lecture

Signals and systems that process them exist in everyday life and in many applications that we commonly think of, and also ones that we don't.

Communications, cell phones, SONAR, Weather Forecasts, Biomedical Engineering, Physical Sciences, Multimedia, Control Systems

From our previous lecture, we know that signals can be either continuous time signals (Independent Variable is a continuous variable), or a discrete time signal (Independent Variable is an integer value).

Implementation Platforms: Continuous Time (CT) hardware: circuits, electronics

Digital Hardware: microprocessors, ASIC's, FPGAs

Digital Software implemented on general purpose digital systems: C++, Java, MATLAB

Digital Software implemented on microprocessors (DSP chips): assembly language, cross compiled C-programs

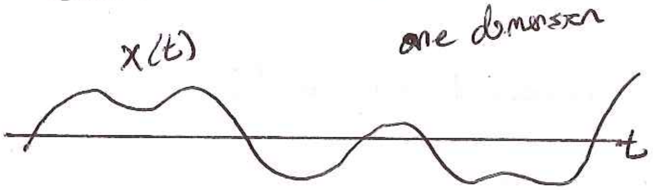
We will deal with a specific class of systems: linear, time-Invariant Systems (LTI) and will look at signal representations in the time domain + frequency domain, learn relationships between the two, and how to convert one to another.

Signals

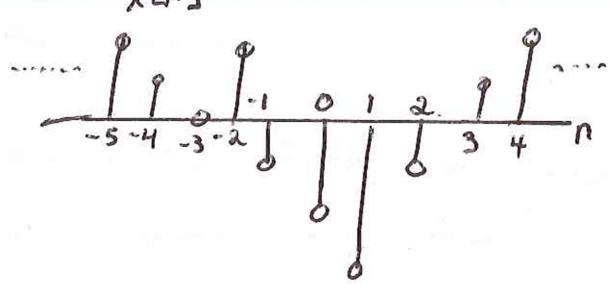
Different than digital

Continuous-Time (CT)

one dimension



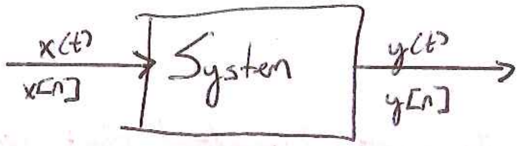
Discrete-Time (DT)



Images are two dimensional brightness (horizontal, vertical)

also multi-dimensional signals $x[n,m]$

Systems



Time Domain

$x(t)$
$x[n]$

LTI

linear time-invariant

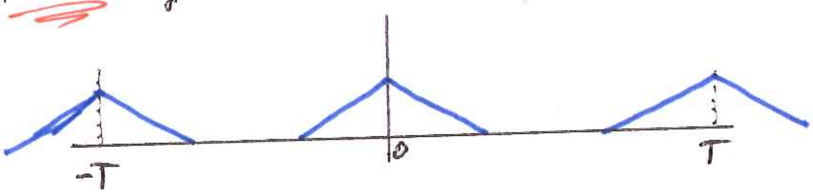
Frequency Domain

Fourier Transform
Laplace Transform
z- transform

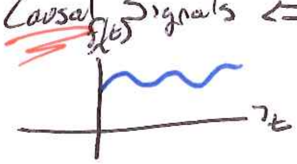
Interconnections of signals

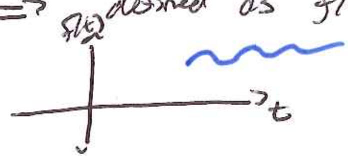
Parallel, Serial, or feedback

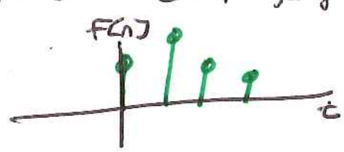
- Periodic Signals $\Longleftrightarrow$ Mathematically defined as a function $f(t) = f(t+T)$ T = period
  $f(t \pm nT)$



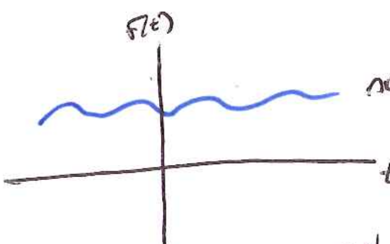
- Causal Signals $\Longleftrightarrow$ defined as $f(t) = 0$ $t < 0$, signal begins at or after 0







$f[n] = 0, n < 0$



non-causal, values before $t=0$

"anti-causal" only signals before $t=0$, opposite of causal

All anticausal systems are non-causal by definition

What about periodic signals? are the causal, non-causal, anti-causal?

Periodic signals never end and are infinite length.

Stability

$|f(t)| < \infty$, all t

$|f[n]| < \infty$, all n

$f(t) = e^{-t}$ non-stable, t exists at ∞

↳ blows up!

$f(t) = t^2$ unstable

$\infty = \infty$

$f(t) = 1$, stable

$f(t) = \cos(t)$, stable

$f(t) = \sin(t)$, stable

Quantify Signals

Energy = Power over time

- Instantaneous signal Power : $p(t) = |x(t)|^2 \qquad p[n] = |x[n]|^2$
- Signal Energy : $E(t_0, t_1) = \int_{t_0}^{t_1} |x(t)|^2 \, dt \qquad E(n_0, n_1) = \sum_{n=n_0}^{n_1} |x[n]|^2$
- Average Signal Power $P(t_0, t_1) = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} |x(t)|^2 \, dt$

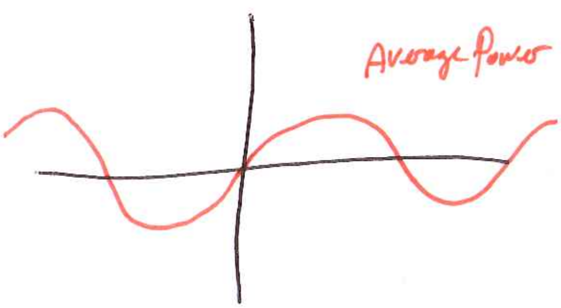
$$P(n_0, n_1) = \frac{1}{n_1 - n_0 + 1} \sum_{n=n_0}^{n_1} |x[n]|^2$$

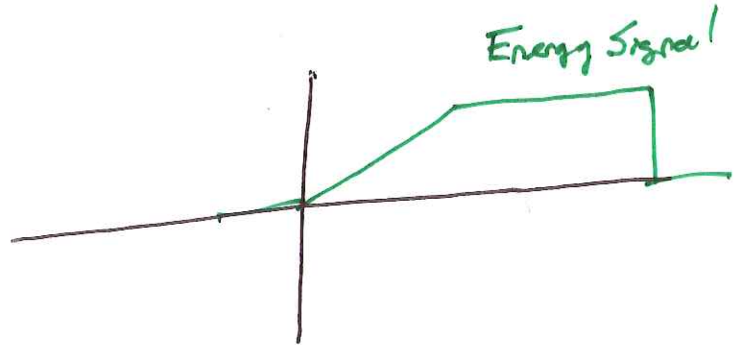
Usual limits are over an infinite Interval:

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \qquad E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \qquad P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- Some signals have infinite power, energy, or both.
- Energy signals if $E_\infty < \infty$
- Power signal if $0 < P_\infty < \infty$
- A signal can be an energy signal, a power signal, or neither
- A signal cannot be both an energy and a power signal.
- If $E_\infty < \infty \Rightarrow P_\infty = 0$
- Signals with finite average power have infinite energy $P_\infty > 0 \Rightarrow E_\infty = \infty$





~~Signal Transformations~~

Two Main Concepts

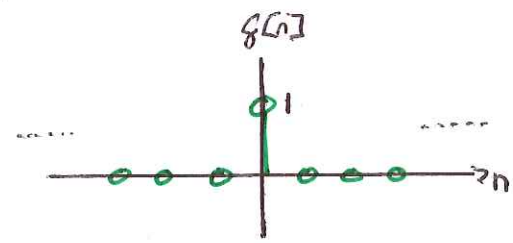
- Signals may be represented as a linear combination of basic signals
- Many systems can be described + analyzed in terms of their responses to basic signals.

Basic Signals

Discrete Time Signals

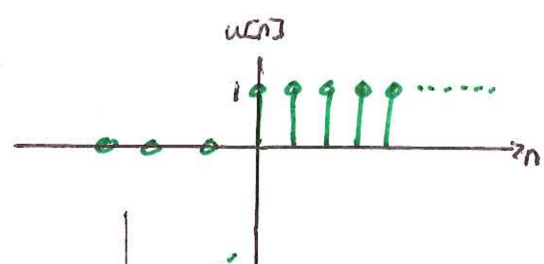
- Impose (or unit impulse) or Delta

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n = 0 \end{cases}$$



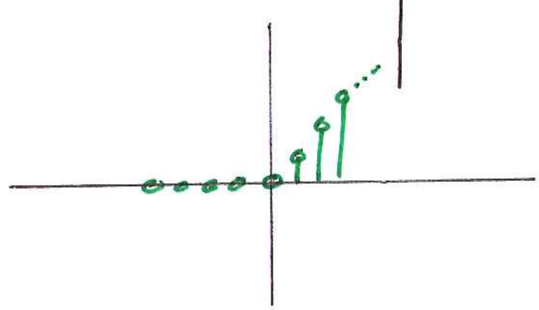
- Step Function

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

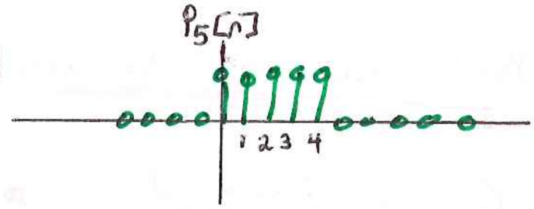


- Ramp Function

$$r[n] = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



■ Pulse (of length N) $\quad p_N[n] = \begin{cases} 1 & n = 0, 1, \cdots N-1 \\ 0 & \text{otherwise} \end{cases}$



Euler's Identities For real and complex sinusoids

$$e^{jx} = \cos x + j \sin x$$

$$\cos x = \frac{1}{2} e^{jx} + \frac{1}{2} e^{-jx}$$

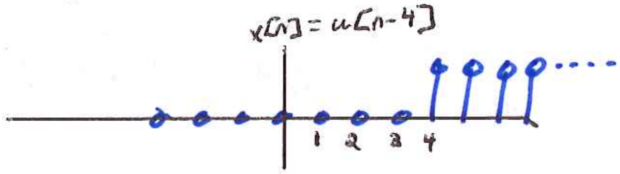
$$\sin x = \frac{1}{2j} e^{jx} - \frac{1}{2j} e^{-jx}$$

Basic Signal Operators

■ Time Shift (delay) : $x(t - t_0) \rightarrow x[n - n_0]$ $\quad$ if $n_0 > 0$, shift to right
if $n_0 < 0$, shift to left

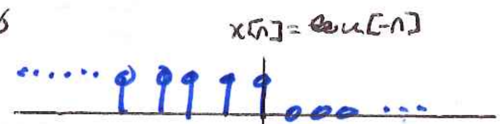
Ex. if $u[n]$, plot $x[n] = u[n-m]$ for $m = 3$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \Rightarrow u[n-m] = \begin{cases} 1 & n - m \geq 0 \\ 0 & n - m < 0 \end{cases} \text{ OR } u[n-m] = \begin{cases} 1 & n \geq m \\ 0 & n < m \end{cases}$$



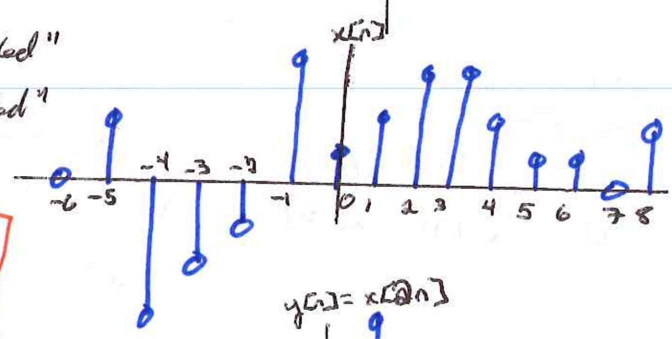
■ Fold / Time Reversal → a reflection of a signal about $n = \phi$

Plot $x[n] = u[-n]$



$x[n] = u[n]$ is "right sided"

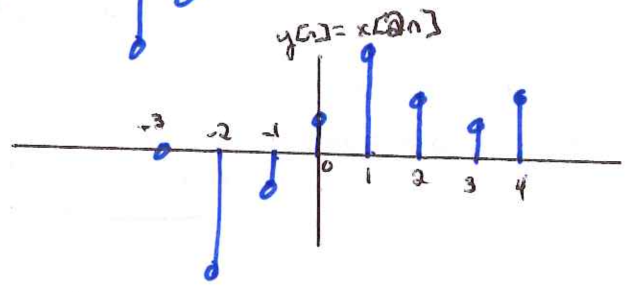
$x[n] = u[-n]$ is "left sided"



■ Time Scaling $x[an]$

- if $a > 1$, signal compresses
- ~~if~~ if $0 < a < 1$ signal stretches/expands

in discrete time, lose intermediate samples!



Basic Properties can be combined!

Continuous Time Signals

- Step $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

- Ramp $r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$

- Pulse $p_T(t) = \begin{cases} 1 & |t| \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$

- Impulse $\delta(t)$, impulse of unit area

  For a very small positive constant ϵ,
  $$\int_{-\epsilon}^{\epsilon} \delta(t)\,dt = 1$$

Signal Operators

- Shift (delay): $x(t) \longrightarrow x(t-t_0)$
- Fold: $x(t) \longrightarrow x(-t)$
- Fold+Shift: $x(t) \longrightarrow x(\tau-t)$
- Multiplication + Addition: Point-by-Point in time
- Differentiation + Integration: $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau = \int_{0}^{\infty} x(t-\tau)\,d\tau$

$$\frac{d}{dt}\, c \cdot u(t) = c\,\delta(t)$$

$$\int_{-\infty}^{t} c\,\delta(\tau)\,d\tau = c\,u(t)$$

Even + Odd Signals

even: $x(t) = x(-t)$

odd: $x(t) = -x(-t)$

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

$$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

$\cos(t)$ = even signal

$\sin(t)$ = odd signal

any signal can be written as the sum of an odd + even signal

Linearity

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Say we have $x_1(t) \to y_1(t)$

$x_2(t) \to y_2(t)$

A system is linear only if $a_1x_1(t) + a_2x_2(t) \longrightarrow a_1y_1(t) + a_2y_2(t)$

Time Invariance

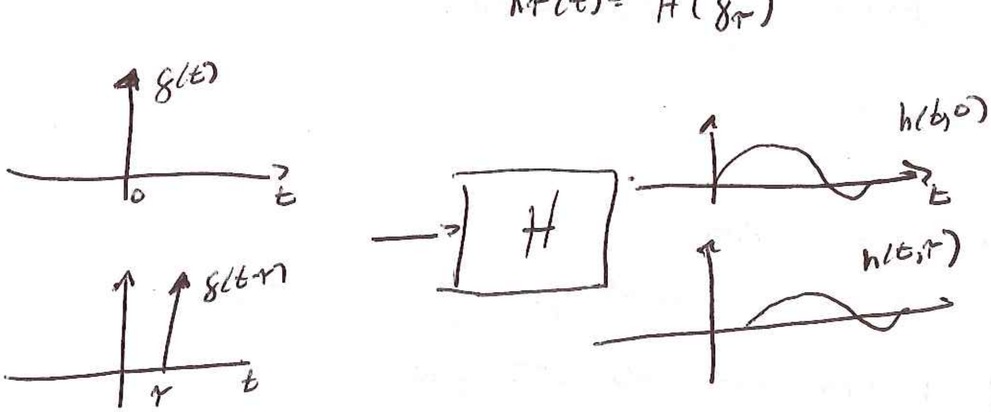
a system is time invariant if and only if $x[n] \to y[n]$ implies that $x[n-n_0] \longrightarrow y[n-n_0]$

Circuits with no energy stored are time invariant.

$\boxed{\text{Impulse Response}}$

The impulse response of a linear system $h_\tau(t)$ is the output of the system at time t to an impulse at time τ.

$$h_\tau(t) = H(\delta_\tau)$$



Integral of impulse

$$\int_a^b \delta(t)\,dt = \begin{cases} 1, & a<0<b \\ 0, & o.w. \end{cases}$$

$$\int_a^b \delta(t-T) = \begin{cases} 1, & a<T<b \\ 0, & o.w. \end{cases}$$

Sifting Property - evaluate the integral of a function multiplied by an impulse at the origin.

$$\int_{-\infty}^{\infty} \delta(t)\cdot f(t)\,dt \Rightarrow$$

because impulse is zero everywhere else

$$\int_{-\infty}^{\infty} \delta(t)\, f(0)\,dt$$

$$\therefore \int_{-\infty}^{\infty} \delta(t)\cdot f(t)\,dt = \int_{-\infty}^{\infty} \delta(t)\, f(0)\,dt$$

$$= f(0)\cdot \int_{-\infty}^{\infty} \delta(t)\,dt$$

$$= f(0)$$

in general $\int_a^b \delta(t)\cdot f(t)\,dt = f(0)\cdot \int_a^b \delta(t)\,dt$

$$= \begin{cases} f(0), & a<0<b \\ 0, & o.w. \end{cases}$$

Thus, because $\delta(t-T)$ is 0 except at T, we can show: $\int_a^b \delta(t-T)\cdot f(t)\,dt = \begin{cases} f(T) & a<T<b \\ 0, & o.w. \end{cases}$

This results in the sifting property! $\int_a^b \delta(t-T)\cdot f(t)\,dt = \begin{cases} f(T) & a<T<b \\ 0, & o.w. \end{cases}$

Discrete time: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

• Discrete Time Convolution

Any discrete-time input signal $x[n]$ can be expressed as a sum of scaled unit impulses.

$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

• By linearity + time invariance, the output of the system is the scaled sum of outputs due to each unit impulse

• $x[n] \longrightarrow y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

$*$: Convolution operator

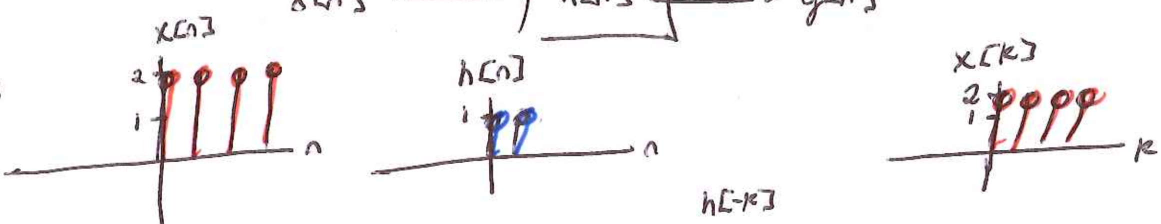
Discrete time convolution sum!

if we know $h[n]$, we can calculate $y[n]$ for any input $x[n]$

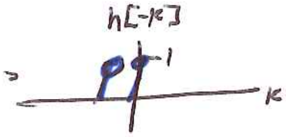
$h[n]$ completely characterizes the LTI system.

$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$

Example:



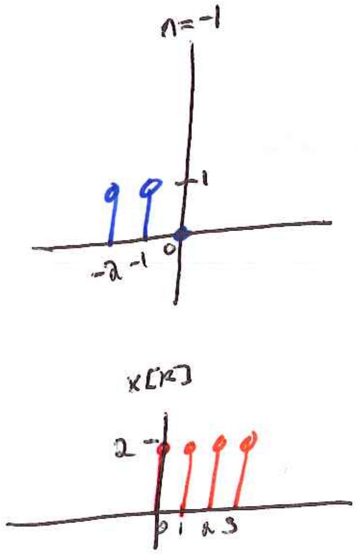
① Flip impulse response $h[k] \to h[-k] \to$



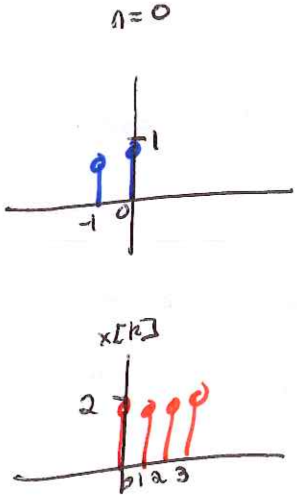
② Drag/Slide $h[-k]$ through $x[k]$ over n, yielding $h(-(k-n))$

and multiply pointwise by $x \to$ yielding $x[k]\cdot h[n-k]$

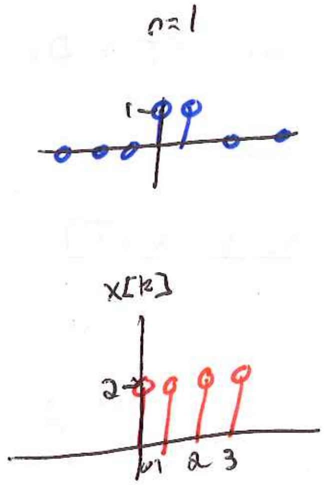
such that $y[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]$



$y[-1] = \emptyset$



$y[0] = 1 \times 2 = 2$



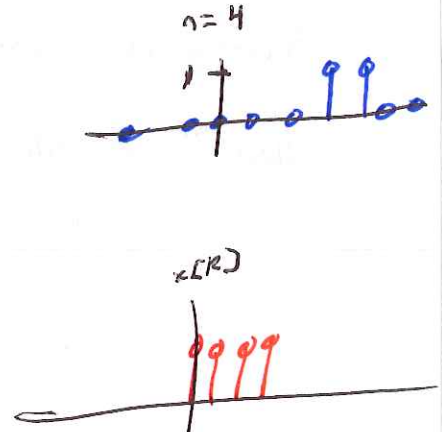
$y[1] = 2\times1 + 2\times1 = 4$

$y[2] = 2\times1 + 2\times1 = 4$

$y[3] = 2\times1 + 2\times1 = 4$

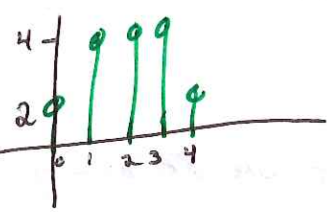
$y[4] = 2\times1 + 0 = 2$

$y[5] = \emptyset$



$y[4] = 2\times1 = 2$

$\therefore y[n] =$



Sampling + Quantization

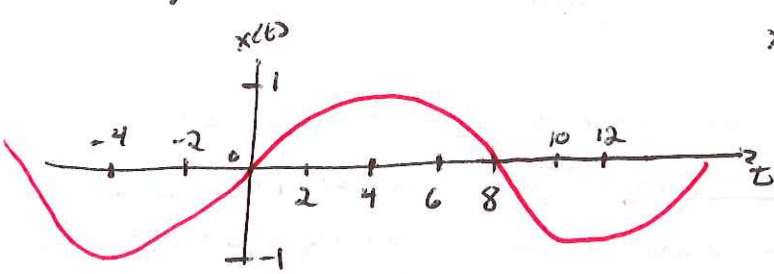
- Analog signals are discretized through sampling. When sampled, they are then quantized into digital signals.

  We convert from analog to digital for storage, + must quantize to specific discrete levels to fit into a finite amount of memory.

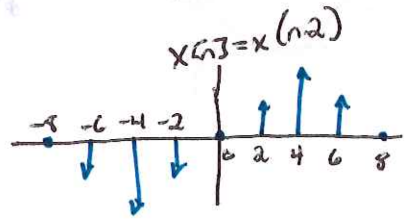
  More quantized levels, the more fidelity + resolution we get.

Sampling converts $x(t) \to x[n]$, much like a $\delta(t)$ sifting operator

$$x[n] = x(n \cdot 2)$$



$T_s = \frac{1}{f_s}$



Sampling is a bridge between CT + DT.

How often should we sample in order to NOT lose any information?

$$\text{if } x[n] = x(nT_s)$$

T_s is interval between Samples

$f_s = \frac{1}{T_s}$ is sampling rate

Fourier Transform next week will help with the

Nyquist - Sampling theorem that states we need to sample a signal at twice its highest Frequency, $f_s = 2f_{signal}$ to be able to full reconstruct it.

Sampling at the Nyquist Rate will allows us to remove ~~the~~ aliasing artifacts from sampling.

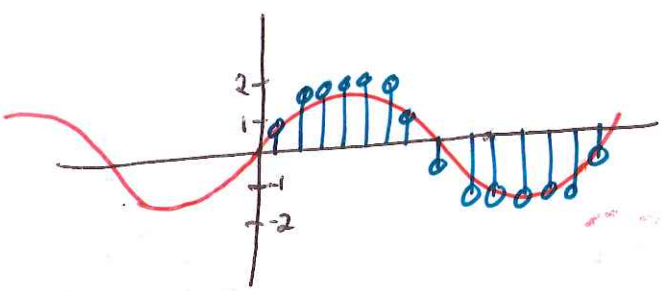
After sampling, we quantize the samples into digital sets or quantities

Quantization levels are usually $N = 2^R$ where N is the number of levels + R is the # of bits.

DAC = Digital-to-Analog Converter for transmission

2 bit Quantizer $2^2 = 4$ levels

ADC = Analog-to-Digital Converter for sampling at discrete quantized levels.



If we sample a signal ranging from 0 to 4 Volts with Quantized bins of 0.5 volts, we have 8 amplitude levels: 0.5, 1, 1.5, 2, 2.5, 3, 3.5, + 4 volts

$\therefore 2^3 = 8 \rightarrow$ results in a 3-bit ADC/Quantizer

Karnaugh Map - Recap

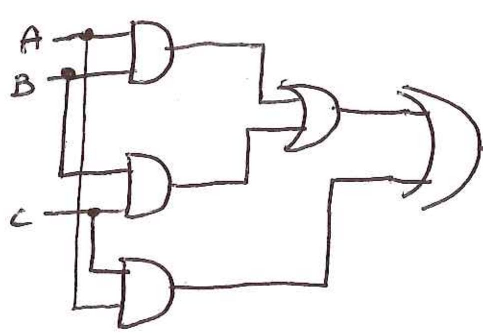
$$\bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$

C \ AB	00	01	11	10
0	0	0	1	0
1	0	1	1	1

1's represent product term

$$AB + AC + BC$$

$$AB + BC + AC$$



$f(A,B,C) = AB + BC + AC$